The L-Shaped method: A tutorial

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Outline

Introduction

Benders decomposition and L-Shaped method

Industrial example
Two-stage stochastic programs

Example 1: The Generator Maintenance Scheduling Problem (GMSP)
Two-stage stochastic programs (cont.)

Example 2: Stochastic Vehicle Routing (SVRP)

Stage 1
Given a set of customers with uncertain demands, plan the route

Stage 2
Visit the customers, observe the demand and take recourse action
Extended formulation

In general,
Stage 1: Non-anticipative decisions
Stage 2: \( \forall \text{ scenario } \omega \in \Omega, \text{ uncertain parameters } \Rightarrow \text{ recourse decisions.} \)
Extended formulation

In general,
Stage 1: Non-anticipative decisions
Stage 2: ∀ scenario \( \omega \in \Omega \), uncertain parameters \( \Rightarrow \) recourse decisions.

If the problem is linear with \( W = \vert \Omega \vert \) scenarios, its extensive form is

\[
\begin{align*}
\min_{x_\omega} & \quad f^T y + \sum_{\omega \in \Omega} p_{\omega} c^T x_\omega \\
\text{s.t.} & \quad B_\omega y + Ax_\omega \geq \xi_\omega, \forall \omega \in \Omega, \\
& \quad x_\omega \geq 0, \forall \omega \in \Omega, \\
& \quad y \in S.
\end{align*}
\]
Extended formulation

In general,

Stage 1: Non-anticipative decisions
Stage 2: \( \forall \) scenario \( \omega \in \Omega \), uncertain parameters \( \Rightarrow \) recourse decisions.

If the problem is linear with \( W = |\Omega| \) scenarios, its extensive form is

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\min_{x_{\omega}} f^T y + \sum_{\omega \in \Omega} p_{\omega} c^T x_{\omega}
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\[
\text{s.t. } B_{\omega} y + A x_{\omega} \geq \xi_{\omega}, \forall \omega \in \Omega,
\]

\[
x_{\omega} \geq 0, \forall \omega \in \Omega,
\]

\[
y \in S.
\]

We want to exploit its dual block angular structure

\[
\begin{pmatrix}
B_1^T & A_1^T \\
B_2^T & A_2^T \\
\vdots & \vdots \\
B_W^T & A_W^T
\end{pmatrix}
\]
Compact representation

This problem can be compactly represented as

$$\min_{y \in S} f^T y + Q(y),$$

where $Q(y)$ is the expected value function

$$Q(y) = E_\omega \{ Q_\omega(y) \},$$
Compact representation

This problem can be compactly represented as

$$\min_{y \in S} f^T y + Q(y),$$

where $Q(y)$ is the expected value function

$$Q(y) = E_\omega \{ Q_\omega(y) \},$$

and for each scenario $\omega$,

$$Q_\omega(y) = \min_{x_\omega} c^T x_\omega$$

s.t. $B_\omega y + Ax_\omega \geq \xi_\omega$

$$x_\omega \geq 0.$$
Benders decomposition: basic idea

Given

\[
\begin{align*}
\min_{x,y} & \quad c^T x + f^T y \\
\text{s.t.} & \quad Ax + By \geq b, \\
& \quad x \geq 0, \\
& \quad y \in S,
\end{align*}
\]

(P)

where \( S \) is a non-convex set.

We fix \( y = \bar{y} \), to obtain the subproblems:

\[
Q(\bar{y}) = \min_{x} c^T x \\
\text{s.t.} \quad Ax \geq b - B\bar{y} \\
& \quad x \geq 0,
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(SP)
Benders decomposition: basic idea

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And we use the subproblem solution to approximate \( Q(y) \).
Graphical representation for 1 variable $y$
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Benders decomposition: feasibility

\[ Q(\bar{y}) = \min_x c^T x \]
\[ \text{s.t. } Ax \geq b - B\bar{y} \]
\[ x \geq 0, \]

To guarantee the feasibility of SP,

\[ \bar{y} \in R = \{ y \in S \mid \exists x \geq 0; Ax \geq b - By \}, \]
\[ = \{ y \in S \mid [b - By]^T u \leq 0 \mid u \geq 0; A^T u \leq 0 \} \]

by Farkas Lemma.
Benders decomposition: feasibility

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by Farkas Lemma.
Since the polyhedral cone \( C = \{u \geq 0; A^T u \leq 0\} \) has a finite number of generators
Benders decomposition: feasibility

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by Farkas Lemma.

Since the polyhedral cone \( C = \{ u \geq 0; A^T u \leq 0 \} \) has a finite number of generators

\[
R = \{ y \in S \mid [b - By]^T u_i^r \leq 0, \forall i \in R \} \tag{1}
\]
Benders decomposition: derivation

\[ P = \min_{x, y} \{ f(y) + c^{T}x \mid Ax \geq b - By; x \geq 0; y \in S \} \]
Benders decomposition: derivation

\[
P = \min_{x,y} \{ f(y) + c^T x \mid Ax \geq b - By; x \geq 0; y \in S \}
\]
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= \min_{y \in R} \{ f(y) + \min_x \{ c^T x \mid Ax \geq b - By; x \geq 0 \}\}\]
Benders decomposition: derivation

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\[ = \min_y \left\{ f(y) + \max_u \left\{ \left[ b - By \right]^T u \mid A^T u \leq c; u \geq 0 \right\} \right\} \]
Benders decomposition: derivation

\[ P = \min_{x,y} \left\{ f(y) + c^T x \mid Ax \geq b - By; x \geq 0; y \in S \right\} \]

\[ = \min_{y \in \mathbb{R}} \left\{ f(y) + \min_{x} \left\{ c^T x \mid Ax \geq b - By; x \geq 0 \right\} \right\} \]

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\[ = \min_{y \in \mathbb{R}} \left\{ f(y) + \max_{p \in \mathcal{P}} \left\{ [b - By]^T u^p \right\} \right\} \]
Benders decomposition: derivation

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\[ = \min_{y \in \mathbb{R}} \left\{ f(y) + \max_{p \in \mathcal{P}} \left\{ [b - By]^T u^p \right\} \right\} \]

\[ = \min_{y \in \mathbb{R}} \left\{ f(y) + w^{SP} \mid w^{SP} \geq [b - By]^T u^p \quad \forall p \in \mathcal{P} \right\} \]
Benders decomposition

The Relaxed Master Problem (RMP) results from relaxing dual constraints (optimality cuts and feasibility cuts) in MP. The most violated constraints can be iteratively computed and included from the dual information of the subproblem SP,
Benders decomposition

The Relaxed Master Problem (RMP) results from relaxing dual constraints (optimality cuts and feasibility cuts) in MP. The most violated constraints can be iteratively computed and included from the dual information of the subproblem SP,

\[ Q(\bar{y}) = \min_{x} \ c^T x \]

subject to
\[ Ax \geq b - B\bar{y} \]
\[ x \geq 0, \]
Example: Generator Maintenance Scheduling

**Relaxed Master Problem (RMP)**

- Maintenance scheduling
- + optimality cuts ($u_p$)
- + feasibility cuts ($u_r$)
- Combinatorial optimization problem

**Candidate maintenance schedules ($y^*$)**

**Subproblems (SP)**

- Hydro power operation
- - Fixed maintenance schedule ($y$)
- Stochastic linear program decomposed by scenario

**Dual solutions ($u_p$)**

**Extreme directions ($u_r$)**
Benders algorithm

Solution algorithm
Benders algorithm

Solution algorithm

1. Solve the relaxed MP
Benders algorithm

Solution algorithm

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   - If unfeasible, STOP. P is also unfeasible.
Benders algorithm

Solution algorithm

1. Solve the relaxed MP
   - If unfeasible, STOP. P is also unfeasible.
   - Otherwise, read solution $y^k$ and optimal value $z^k$ ($LB$).
Benders algorithm

Solution algorithm

1. Solve the relaxed MP
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2. Solve $SP_\omega(y^k) \forall \omega \in \Omega$. 
Benders algorithm

Solution algorithm

1. Solve the relaxed MP
   ▶ If unfeasible, STOP. P is also unfeasible.
   ▶ Otherwise, read solution $y^k$ and optimal value $z^k$ ($LB$).

2. Solve $SP_\omega(y^k)$ $\forall \omega \in \Omega$.
   ▶ If unfeasible, read extreme dual ray $u^r_\omega(y^k)$. 

4. If stopping criteria are met ($UB - LB \leq \epsilon$; time; iterations), STOP.

5. Otherwise, compute cuts and add cuts to MP. Go to step 1.
Benders algorithm

Solution algorithm

1. Solve the relaxed MP
   - If unfeasible, STOP. P is also unfeasible.
   - Otherwise, read solution $y^k$ and optimal value $z^k$ (LB).
2. Solve $SP_\omega(y^k) \ \forall \ \omega \in \Omega$.
   - If unfeasible, read extreme dual ray $u^r_\omega(y^k)$.
   - If optimal, read dual solution $u^p_\omega(y^k)$ and solution value $z_\omega(y^k)$.
3. Use $z_\omega(y^k)$ to compute $UB$
4. If stopping criteria are met ($UB - LB \leq \epsilon$; time; iterations), STOP.
5. Otherwise, compute cuts and add cuts to MP. Go to step 1.
Toy problem example

Implementation in C++ with Xpress BCL.

Solving the problem
Minimize 2*x1 + 3*x2 + 2*y1 + y2
St:
x1 + 2*x2 + y1 + 2*y2 >= 5;
x1, x2 >= 0;
y1, y2 : Binary

Solution for the complete problem using the MIP solver
x1 = 0.4
x2 = 0.8
y1 = 1
y2 = 1
Optimal value = 6.2

********* Starting Benders Procedure *********
N.Cut L.Bound U.Bound Y1 Y2 x1 x2

0.00 8.80 0.00 0.00 2.60 1.20
Optimality_cut_0: Z + 2.2*Y1 + 3.4*Y2 >= 8.8

1 6.20 6.20 1.00 1.00 0.40 0.80
Optimal solution found after 1 cut(s).

Code available at: http://osg.polymtl.ca
Parallel implementation

Server
- Read problem parameters
- Allocate scenarios to nodes
- Create master problem
- Compute lower bound
- Solve master problem
- Check stopping criteria
- Compute and add cuts

Compute nodes
- Subproblem parameters
- Create subproblems
- Master problem solution
- Fix master problem solutions into subproblems
- Solve subproblems
- Subproblems objective values
- Dual solutions
- Read subproblem dual solutions
- Compute and add cuts
Branch and cut approach

When the Master Problem is difficult to solve, the classical approach can perform poorly. A branch and cut approach can be used to solve a single tree. Whenever an integer solution is found in a node of the tree, the subproblems are solved to check optimality and feasibility. Cuts are added if necessary, if necessary.
Integer L-Shaped

If the subproblems are integer, dual variables are unavailable. Cuts can be computed from bounds and objective values of the subproblems (Laporte and Louveaux, 1993)

\[ w_{SP} \geq (Q(\bar{y}) - LB_{sp}) \left( \sum_{(m,t) \in \mathcal{A}} (y_{mt} - 1) - \sum_{(m,t) \notin \mathcal{A}} (y_{mt}) \right) - Q(\bar{y}) \]  (2)
Application example: Generator Maintenance Scheduling Problem

How to use a decomposition approach to efficiently produce maintenance schedules that minimize the maintenance and operation costs in hydropower systems?

A decomposition approach for the hydropower operation and maintenance scheduling problem

Jesus Rodriguez, Miguel Anjos, Charles Audet, and Pascal Côté

Hydropower provides 97% of the renewable electricity in Canada and 46% in the USA.

Maintenance outages

- Reduces service disruptions
- Prevents costly breakdowns

Temporal and spatial interdependencies

Nonlinear power functions

Uncertain Inflows

\[ P_t = f(Q_t, h_t, \Omega_t) \]

Figure: Factors that impact the operation of hydropower systems
HGMSP: Problem definition

Hydropower maintenance scheduling problem

Minimize Maintenance and Operation cost

Subject to:

- Maintenance constraints
  - Completion of maintenance tasks
  - Time windows
  - Maximum number of outages
  - Maintenance capacity
- Hydropower operation constraints
  - Mass balance
  - Power balance
  - Power production function
  - Bounds on reservoir levels and discharges.

Maintenance & Operation Plan
$x_\omega$: hydropower operation variables of scenario $\omega$,

$y$: binary variables representing the beginning of maintenance activities,

$z$: binary variables representing the number of active generators,

$$\min_{x_\omega, y} \sum_{\omega \in \Omega} \varphi_\omega c^T x_\omega + f^T y$$

s.t.

$$A_1 x_\omega + C_1 z \geq b_{1\omega}, \quad \forall \omega \in \Omega$$

$$A_2 x_\omega \geq b_{2\omega}, \quad \forall \omega \in \Omega$$

$$B_3 y + C_3 z = b_3$$

$$B_4 y \geq b_4$$

$$C_5 z \geq b_5$$

$$x_\omega \geq 0, \quad \forall \omega \in \Omega$$

$$y \in \{0, 1\}$$

$$z \in \{0, 1\}$$
fixing $z$ splits the problem into operation subproblems for each scenario $\omega \in \Omega$,

$$Q_w(\bar{z}) = \min_{x_\omega} c^T x_\omega$$

s.t.  

$$A_1 x_\omega \geq b_{1\omega} - C_1 \bar{z} \perp \lambda_{1\omega}$$  
$$A_2 x_\omega \geq b_{2\omega} \perp \lambda_{2\omega}$$  
$$x_\omega \geq 0$$

and a master problem,

$$\min_{w^{SP}, y, z} w^{SP} + f^T y$$

s.t.  

$$B_3 y + C_3 z = b_3$$  
$$B_4 y \geq b_4$$  
$$C_5 z \geq b_5$$  
$$y \in \{0, 1\}$$  
$$z \in \{0, 1\}$$

$$w_{SP} - f^T y + \sum_{\omega \in \Omega} \varphi_{\omega} \lambda_{1p}^{\omega} [C_1 z] \geq \sum_{\omega \in \Omega} \varphi_{\omega} (b_{1\omega}^T \lambda_{1p} + b_{2\omega}^T \lambda_{2p})$$  
$$\forall p \in \mathcal{P}$$
Conclusions

- A careful analysis of the problem structure is necessary for a good choice of the solution strategy.
- Explore alternative formulations. The stronger the formulation, the better the performance of the decomposition method.
- The intuitive partitioning of the problem is not necessarily the most effective one.
- Straightforward implementation of decomposition methods can be disappointing. Customization and acceleration strategies are necessary.
- If subproblems are linear, you can benefit from open source solvers.
Slides and code of the example will be available on the website of Optimization for Smart Grids (OSG)

http://osg.polymtl.ca

Thank you!

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